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ON CONSERVATION OF L_2 NORM OF THE SOLUTION BY THE FINITE DIFFERENCE SCHEMES FOR ONE FAMILY OF EVOLUTION EQUATIONS

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Abstract

One family of evolution equations is considered, which is characterized by the conservation of L_2 norm of its solution. Corresponding finite difference schemes are investigated and it is proved that two-layer finite difference scheme does not conserve L_2 norm of the solution whereas the three-layer scheme conserves one analogues of L_2 norm. Using obtained results one particular evolution equation is studied.

Key words and phrases: Korteweg-de Vries equation, Finite difference scheme, Conservation of L_2 norm.

AMS subject classification: 35Q53, 65N06.

Introduction

The work is devoted to properties of finite difference schemes for one family of evolution equations. This family is characterized by the conservation of L_2 norm of its solution and naturally rises the problem: which finite difference schemes corresponding to these equations inherit this property and which does not.

This problem is studied in [1] on example of Korteweg - de Vries equation, where it was proved that the simplest two-layer scheme does not conserve L_2 norm of the solution whereas the three-layer scheme conserves one analogues of L_2 norm.

We generalized this result on larger class of nonlinear equations for $n-$ dimensional space case.

The research consists of two parts: in the first part results in general form are presented. In the second part for illustration of the method we consider one particular family of evolution equations [2], [3].

1. Conservation of L_2 Norm by the Two and Three Layer Finite Difference Schemes

Let us consider the set of real functions H on domain $\Omega = [0,1] \times [0,1] \times \cdots \times [0,1]$

$$H = \{u | u : \Omega \to R\} \tag{1.1}$$

and define a scalar product and norm on H as

$$(u, v) = \int_{\Omega} uv dx,$$

$$\|u\| = \sqrt{(u, u)}.$$

Let us consider the set of elements of H

$$\{u(t)\,|u(t)\in H\,,\quad t\in[0,T]\},$$

which satisfies the conditions:

$$\frac{d}{dt}u(t)+Lu(t)=f(t),\qquad 0\leqslant t\leqslant T, \tag{1.2}$$

$$u(0)=u_0,\qquad u_0\in H,$$

where

$$L:H\to H,$$

$$f:[0,T]\to H.$$

According to the definition [4], $u(t)\in H$ is a solution of (1.2), if

$$\left\|\frac{u(t+\Delta t)-u(t)}{\Delta t}+Lu(t)-f(t)\right\|\to 0,$$

when $\Delta t\to 0$, $0<t<T$.

The corresponding homogeneous equation of (1.2) has the following form

$$\frac{d}{dt}u(t)+Lu(t)=0,\qquad 0\leqslant t\leqslant T \tag{1.3}$$

and if operator L satisfies the condition

$$(Lu,u)=0, \tag{1.4}$$

we will derive, that

$$\frac{d}{dt}\|u(t)\|=0, \tag{1.5}$$

i. e. $\|u(t)\|$ value is conserved. Often this value has the meaning of energy and this property of the solution is called a conservation of energy [5], [6].

Below we construct the finite difference schemes for (1.3) and research whether difference analog of (1.5) takes place.

Let

$$\overline{\omega}_\tau=\{t_k=\tau k,\quad k=0,1,...,N;\quad \tau N=T\},$$

$$\overline{\omega}_{h_i}=\{x_k=kh_i,\quad k=0,1,...,N_i;\ \ h_iN_i=1\}\quad i=\overline{1,n},$$

$$\Omega_h=\overline{\omega_{h_1}}\times\overline{\omega_{h_2}}\times\cdots\times\overline{\omega_{h_n}},$$

$$H_h=\{z\,|z:\Omega_h\to R\},$$

define the scalar product $(\cdot,\cdot)_h$ on H_h, which is the difference analog of (1.1) and consider $L_h : H_h \to H_h$ operator which is the difference approximation of $L : H \to H$ and which inherits (1.4) property in sense of $(\cdot,\cdot)_h$ scalar product

$$(L_h z, z)_h = 0, \tag{1.6}$$

where $z \in H_h$.

Let us consider the difference analog of the family of one parameter functions $u(t)$ in H_h space

$$\{z_k \,|\, z_k \in H_h\,, \quad k = 0, 1, ..., N\}$$

and construct two-layer finite difference scheme for equation (1.3)

$$z_t + L_h z = 0, \tag{1.7}$$

where

$$z = z_k, \quad z_t = z_{t,k} = \frac{z_{k+1} - z_k}{\tau}$$

and L_h is the difference analog of L which has property (1.6).

If we multiply difference equation (1.7) on $2\tau z_k$ and take into account that

$$2\tau z_k = \tau\,(z_{k+1} + z_k) - \tau\,(z_{k+1} - z_k)\,,$$

we obtain

$$\|z_{k+1}\|_h^2 - \|z_k\|_h^2 - \|z_{k+1} - z_k\|_h^2 = 0,$$

$$\|z_{k+1}\|_h^2 - \|z_k\|_h^2 - \tau^2\,\|z_{t,k}\|_h^2 = 0,$$

$$\|z_{k+1}\|_h^2 = \|z_0\|_h^2 + \tau^2 \sum_{i=0}^{k} \|z_{t,i}\|_h^2\,,$$

i. e. scheme (1.7) does not conserve $\|z\|_h$ value.

Now let us consider the three-layer finite difference scheme which corresponds to equation (1.3)

$$z_{\overset{0}{t}} + L_h z = 0, \tag{1.8}$$

where

$$z_{\overset{0}{t}} = \frac{z_{k+1} - z_{k-1}}{2\tau}$$

and L_h operator is the difference analog of operator L, which satisfies condition (1.6).

If we multiply (1.8) on $2\tau z_k$, we will obtain, that

$$\left(\frac{z_{k+1} - z_{k-1}}{2\tau}, 2\tau z_k\right)_h = 0,$$

so

$$(z_{k+1}, z_k)_h = (z_{k-1}, z_k)_h,$$

i. e. the energetic value $(\widehat{z}, z)_h$ is conserved, where $z = z_h$, $\widehat{z} = z_{k+1}$.

2. Conservation of L_2 Norm for One Family of Evolution Equations

In this section we illustrate obtained results for the following family of evolution equations of type (1.3)

$$\frac{\partial u}{\partial t} + Lu = 0, \tag{2.1}$$

where

$$Lu = \left(\sum_{i=1}^{m} \alpha_i u^i\right)\left(\sum_{k=1}^{n} \beta_k \frac{\partial u}{\partial x_k}\right) + \sum_{j=1}^{n}\sum_{k=1}^{n} \gamma_k^j \frac{\partial^3 u}{\partial x_j^2 \partial x_k},$$

$$\alpha_i, \beta_k, \gamma_k^j \in R, \quad i, j, k = 1, ..., n,$$

$$u : \Omega \times [0, T] \to R,$$

$$\Omega = [0, X_1] \times [0, X_2] \times \cdots \times [0, X_n],$$

R denotes the set of real numbers and the solution of (2.1) satisfies the following periodic boundary conditions

$$\frac{\partial^{|\nu|} u(x_1, x_2, ..., x_{k-1}, 0, x_{k+1}, ..., x_n, t)}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_n^{\nu_n}} = \frac{\partial^{|\nu|} u(x_1, x_2, ..., x_{k-1}, 1, x_{k+1}, ..., x_n, t)}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_n^{\nu_n}}, \tag{2.2}$$

when $k = 1, ..., n$ and $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\sum\limits_{i=1}^{n} \nu_i \leqslant 2$.

Let H_0 be the subspace of Hilbert space H, elements of which satisfy boundary conditions (2.2).

It is obvious, that if $u, v \in H_0$, then

$$\left(\frac{\partial u}{\partial x_i}, \nu\right) = -\left(u, \frac{\partial \nu}{\partial x_i}\right), \tag{2.3}$$

as

$$\int_{\Omega} \left(\frac{\partial u}{\partial x_i}\nu + u\frac{\partial \nu}{\partial x_i}\right) dx = \int_{\Omega} \frac{\partial (u\nu)}{\partial x_i} dx = 0$$

and

$$\left(u^i, \frac{\partial u}{\partial x_i}\right) = 0, \tag{2.4}$$

as

$$\int\limits_\Omega u^i \frac{\partial u}{\partial x_i} dx = \int\limits_\Omega \frac{\partial}{\partial x_i}\left(\frac{u^{i+1}}{i+1}\right) dx = 0.$$

From (2.3) we also obtain, that

$$\left(\frac{\partial^3 u}{\partial x_i^2 \partial x_k}, u\right) = 0. \tag{2.5}$$

Indeed,

$$\left(\frac{\partial^3 u}{\partial x_i^2 \partial x_k}, u\right) = \left(\frac{\partial}{\partial x_i}\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right), u\right) = -\left(\frac{\partial^2 u}{\partial x_i \partial x_k}, \frac{\partial u}{\partial x_i}\right) = -\frac{1}{2}\int\limits_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_i}\right)^2 dx = 0.$$

From (2.4) and (2.5) we can conclude, that operator L satisfies condition (1.4).

Let us define the scalar product on set of H_h mesh functions by the following formula

$$(u,\nu) = \frac{1}{2^n}\left(\prod_{i=1}^{n} h_i\right) \sum_{(i_1,\dots,i_n) i_j=1,N_j} \left[\sum_{p_j=0\vee p_j=1 \;\; j=1,N} u_{i_1-p_1,\dots,i_n-p_n}\,\nu_{i_1-p_1,\dots,i_n-p_n}\right].$$

Let us note, that

$$z_{\overset{0}{x}_k \overset{0}{x}_j} = z_{\overset{0}{x}_j \overset{0}{x}_k}, \tag{2.6}$$

as

$$z_{\overset{0}{x}_k \overset{0}{x}_j} = \frac{1}{2h_j}\left[z_{\overset{0}{x}_k,\dots,i_j+1,\dots} - z_{\overset{0}{x}_k,\dots,i_j-1,\dots}\right] =$$

$$= \frac{1}{4h_j h_k}\left[z_{\dots,i_j+1,\dots,i_k+1,\dots} - z_{\dots,i_j+1,\dots,i_k-1,\dots} - z_{\dots,i_j-1,\dots,i_k+1,\dots} + z_{\dots,i_j-1,\dots,i_k-1,\dots}\right] =$$

$$= z_{\overset{0}{x}_j \overset{0}{x}_k}.$$

Let

$$H_{h,0} = \left\{ z^0 \,\middle|\, (z^0 \in H_h)\; z^0_{i_1,i_2,\dots,i_{k-1},i_k+N_k,i_{k+1},\dots,i_n} = z^0_{i_1,i_2,\dots,i_{k-1},i_k,i_{k+1},\dots,i_n}, \quad k=1,\dots,n \right\}$$

and note, that if $z \in H_{h,0}$, then

$$\sum_{(i_1,\dots,i_n)} z_{i_1,\dots,i_n} = \sum_{(i_1+q_1,\dots,i_n+q_n)} z_{i_1+q_1,\dots,i_n+q_n}, \quad q_i \in N_0, \quad i=1,\dots,n. \tag{2.7}$$

As z_0 is periodic, the mesh functions $z^0_{\overset{0}{x}k}$, $z^0_{\overset{0}{x}j\overset{0}{x}j\overset{0}{x}k}$ $k=1,\dots,n$ would be periodic also.

Let us prove, that

$$\left(z_{\overset{0}{x}_i}, y\right)_h = -\left(z, y_{\overset{0}{x}_i}\right)_h. \tag{2.8}$$

Indeed,

$$\left(z_{\overset{0}{x}_k}, y\right)_h = \frac{1}{2^n}\left(\prod_{i=1}^{n} h_i\right) \sum_{(i_1,\ldots,i_n)\ i_j=\overline{1,N_j}} \left[\sum_{p_i=0 \vee p_i=1\ i=\overline{1,N}} z_{\overset{0}{x}_k, i_1-p_1,\ldots,i_n-p_n} y_{i_1-p_1,\ldots,i_n-p_n}\right] =$$

$$= \frac{1}{2^n}\left(\prod_{i=1}^{n} h_i\right) \sum_{(i_1,\ldots,i_n)\ i_j=\overline{1,N_j}} \left[\sum_{p_i=0 \vee p_i=1\ i\neq k} \left(\sum_{p_i=0 \vee p_i=1\ i=k} z_{\overset{0}{x}_k, i_1-p_1,\ldots,i_n-p_n} y_{i_1-p_1,\ldots,i_n-p_n}\right)\right] =$$

$$= -\left(z, y_{\overset{0}{x}_k}\right)_h,$$

as

$$\sum_{p_i=0 \vee p_i=1\ i=k} z_{\overset{0}{x}_k, i_1-p_1,\ldots,i_n-p_n} y_{i_1-p_1,\ldots,i_n-p_n} = z_{\overset{0}{x}_k, i_k-1,\ldots} y_{\ldots,i_k-1,\ldots} + z_{\overset{0}{x}_k,\ldots,i_k,\ldots} y_{\ldots,i_k,\ldots} =$$

$$= \frac{1}{2h_k}\left[z_{\ldots,i_k,\ldots} y_{\ldots,i_k-1,\ldots} - z_{\ldots,i_k-2,\ldots} y_{\ldots,i_k-1,\ldots} + z_{\ldots,i_{k+1},\ldots} y_{\ldots,i_k,\ldots} - z_{\ldots,i_{k-1},\ldots} y_{\ldots,i_k,\ldots}\right] =$$

$$= \frac{1}{2h_k}\left[z_{\ldots,i_k,\ldots} y_{\ldots,i_k-1,\ldots} - z_{\ldots,i_k,\ldots} y_{\ldots,i_k+1,\ldots} + z_{\ldots,i_{k-1},\ldots} y_{\ldots,i_k-2,\ldots} - z_{\ldots,i_{k-1},\ldots} y_{\ldots,i_k,\ldots}\right].$$

In these transformations we used identical equation (2.7).

From (2.8) we obtain that

$$\left(z_{\overset{0}{x}_i}, z\right)_h = 0. \tag{2.9}$$

From (2.6), (2.8) and (2.9) we can conclude that

$$\left(z_{\overset{0}{x}_i \overset{0}{x}_i \overset{0}{x}_j}, z\right)_h = 0. \tag{2.10}$$

Indeed,

$$\left(z_{\overset{0}{x}_i \overset{0}{x}_i \overset{0}{x}_j}, z\right)_h = \left(z_{\overset{0}{x}_i \overset{0}{x}_j \overset{0}{x}_i}, z\right)_h = -\left(z_{\overset{0}{x}_i \overset{0}{x}_j}, z_{\overset{0}{x}_i}\right)_h = 0.$$

Let us consider the difference analog of the operator L

$$L_h z = \left(\sum_{i=0}^{m} \alpha_i \overline{(z^i)}\right)\left(\sum_{k=1}^{n} \beta_k z_{\overset{0}{x}_k}\right) + \sum_{j=1}^{n}\sum_{k=1}^{n} \beta_k^j z_{\overset{0}{x}_j \overset{0}{x}_j \overset{0}{x}_k}, \tag{2.11}$$

where

$$\overline{(z^i)} = \frac{z^i + \sum_{k=0}^{i} (\check{z})^i (\hat{z})^{i-j}}{i+2}, \tag{2.12}$$

$$z = z_k,\ \check{z} = z_{k-1},\ \hat{z} = z_{k+1}$$

and prove, that operator L_h on the set of functions $H_{h,0}$ satisfies conditions (1.6). First let

$$\left(\overline{(z^i)}z_{\overset{0}{x}_k}, z\right) = 0. \tag{2.13}$$

Indeed,

$$\left(\sum_{j=0}^{i}(\check{z})^j(\widehat{z})^{i-j}\right)z_{\overset{0}{x}_k} = \left(\sum_{j=0}^{i}(\check{z})^j(\widehat{z})^{i-j}\right)\frac{\widehat{z}-\check{z}}{2h_k} = \frac{(\widehat{z})^{i+1}-(\check{z})^{i+1}}{2h_k} = \left(z^{i+1}\right)_{\overset{0}{x}_k},$$

$$\left(\overline{(z^i)}z_{\overset{0}{x}_k}, z\right) = \frac{1}{i+2}\left(z^i z_{\overset{0}{x}_k}, z\right) + \frac{1}{i+2}\left(\left(\sum_{j=0}^{i}(\check{z})^j(\widehat{z})^{i-j}\right)z_{\overset{0}{x}_k}, z\right) =$$

$$= \frac{1}{i+2}\left(z^i z_{\overset{0}{x}_k}, z\right) + \frac{1}{i+2}\left(\left(z^{i+1}\right)_{\overset{0}{x}_k}, z\right) = \frac{1}{i+2}\left(z^i z_{\overset{0}{x}_k}, z\right) - \frac{1}{i+2}\left(z^{i+1}, z_{\overset{0}{x}_k}\right) = 0.$$

In the last transformations we have used the definition of scalar product. Due to (2.10) and (2.13), L_h operator, which is defined by formula (2.11), satisfies (1.6). Thus, we have proved, that the corresponding two-layer finite difference scheme of equation (2.1) does not conserve the property of conservation of the energy of the solution, whereas the three-layer finite difference scheme conserves one analog of the energy of the solution, namely $(\widehat{z}, z)_h$ value.

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